

# Spatial Correlations in Chaotic Eigenfunctions

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## Abstract

At short distances, energy eigenfunctions of chaotic systems have spatial correlations that are well described by assuming a microcanonical density in phase space for the corresponding Wigner function. However, this is not correct on large scales. The correct correlation function is in turn needed to get the correct formula for the root-mean-square value of the off-diagonal matrix elements of simple observables, and for the fluctuations in the diagonal elements.

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My focus in this talk will be on the statistical properties of the quantum energy eigenfunctions in classically chaotic systems. Twenty years ago, Berry [1] conjectured that these eigenfunctions could be treated as gaussian random variables, and that the spatial correlations would be those following from taking the expected value of the Wigner density for each eigenfunction to be microcanonical. More specifically, the probability that the actual eigenfunction  $\psi_\alpha(\mathbf{q})$  for energy  $E = E_\alpha$  (the corresponding energy eigenvalue) is between  $\psi(\mathbf{q})$  and  $\psi(\mathbf{q}) + d\psi(\mathbf{q})$  for all coordinate points  $\mathbf{q}$  is given by

$$P(\psi|E) \propto \exp \left[ -\frac{\beta}{2} \int d^f q \int d^f q' \psi^*(\mathbf{q}) K(\mathbf{q}, \mathbf{q}'|E) \psi(\mathbf{q}') \right] \mathcal{D}\psi . \quad (1)$$

If the system is time-reversal invariant, the eigenfunctions are real,  $\beta = 1$ , and the measure is  $\mathcal{D}\psi = \prod_{\mathbf{q}} d\psi(\mathbf{q})$ ; if it is not, the eigenfunctions are complex,  $\beta = 2$ , and the measure is  $\mathcal{D}\psi = \prod_{\mathbf{q}} d\text{Re } \psi(\mathbf{q}) d\text{Im } \psi(\mathbf{q})$ . In either case, the kernel  $K(\mathbf{q}, \mathbf{q}'|E)$  is the functional inverse of the two-point correlation function

$$C(\mathbf{q}', \mathbf{q}|E) \equiv \langle \psi(\mathbf{q}') \psi^*(\mathbf{q}) \rangle , \quad (2)$$

where the angle brackets denote averaging over  $P(\psi|E)$ . Taking the Wigner density to be microcanonical results in

$$C(\mathbf{q}', \mathbf{q}|E) = \frac{1}{\bar{\rho}(E)} \int \frac{d^f p}{(2\pi\hbar)^f} e^{i\mathbf{p} \cdot (\mathbf{q}' - \mathbf{q})/\hbar} \delta(E - H_W(\mathbf{p}, \bar{\mathbf{q}})) , \quad (3)$$

where  $\bar{\mathbf{q}} = \frac{1}{2}(\mathbf{q} + \mathbf{q}')$  is the midpoint,  $H_W(\mathbf{p}, \mathbf{q})$  is the classical hamiltonian (more specifically, it is the Weyl symbol of the hamiltonian operator), and  $\bar{\rho}(E)$  is the semiclassical density of states,

$$\bar{\rho}(E) = \int \frac{d^f p d^f q}{(2\pi\hbar)^f} \delta(E - H_W(\mathbf{p}, \mathbf{q})) . \quad (4)$$

Berry's conjecture for  $C(\mathbf{q}', \mathbf{q}|E)$  can be tested numerically in two-dimensional billiard systems; in this case,  $H = \mathbf{p}^2/2m$ , and we have  $C(\mathbf{q}', \mathbf{q}|E) = J_0(k|\mathbf{q}' - \mathbf{q}|)$ , where  $E = \hbar^2 k^2/2m$  and  $J_0(x)$  is an ordinary Bessel function. An “experimental” correlation function  $C_{\text{exp}}(\mathbf{s}, D|E)$  can be defined in terms of a particular energy eigenfunction  $\psi_\alpha(\mathbf{q})$  via

$$C_{\text{exp}}(\mathbf{s}, D|E) \equiv \frac{A_B}{A_D} \int_D d^2 q \psi_\alpha^*(\mathbf{q} + \frac{1}{2}\mathbf{s}) \psi_\alpha(\mathbf{q} - \frac{1}{2}\mathbf{s}) , \quad (5)$$

where  $E = E_\alpha$ ,  $A_B$  is the area of the billiard, and  $A_D$  is the area of a domain  $D$  over which the central point  $\mathbf{q}$  is averaged. Early computations [2,3] of  $C_{\text{exp}}(\mathbf{s}, D|E)$  found that it typically resembled its predicted value  $J_0(k|\mathbf{s}|)$ , but with large fluctuations. Later it was realized [4] that these fluctuations were in fact entirely consistent with the expected gaussian fluctuations [5] of  $\psi_\alpha(\mathbf{q})$ . Recent detailed computations [6,7] leave little doubt that (at asymptotically high energies) the eigenfunctions of chaotic billiards are very well described by Berry's conjecture.

Berry's conjecture also follows from other approaches to quantum chaos. The random-matrix analogy [8,9] suggests that the energy eigenstates have a Porter-Thomas distribution

of amplitudes in a suitably chosen basis. For billiards, the natural choice is the momentum basis, and from this starting point one can derive eqs. (1) and (3) [10]. If one adds a white-noise, spatially uncorrelated random potential to the system, and takes the limit where the mean-free path is much larger than the system size, then eq. (1) can be derived via the supersymmetric sigma-model technique [11–16].

However, we cannot expect eq. (3) to be valid when the separation between  $\mathbf{q}$  and  $\mathbf{q}'$  becomes large compared to distance scales which are important classically. This is because the hamiltonian evaluated at just the midpoint  $\bar{\mathbf{q}}$  does not contain enough information about the classical landscape between  $\mathbf{q}$  and  $\mathbf{q}'$ . To find the correct formula for  $C(\mathbf{q}', \mathbf{q}|E)$  when  $|\mathbf{q}' - \mathbf{q}|$  is large [17], we first consider the theory with an added random potential. In this case an expression for  $C(\mathbf{q}', \mathbf{q}|E)$  can be derived explicitly [12]:

$$C(\mathbf{q}', \mathbf{q}|E) = \frac{1}{2\pi i \bar{\rho}(E)} \left[ \bar{G}(\mathbf{q}, \mathbf{q}'|E)^* - \bar{G}(\mathbf{q}', \mathbf{q}|E) \right] . \quad (6)$$

Here  $G(\mathbf{q}', \mathbf{q}|E)$  is the energy Green's function,

$$G(\mathbf{q}', \mathbf{q}|E) = \sum_{\alpha} \frac{\psi_{\alpha}(\mathbf{q}') \psi_{\alpha}^*(\mathbf{q})}{E - E_{\alpha} + i0^+} , \quad (7)$$

$\rho(E)$  is the density of states,

$$\rho(E) = \frac{1}{2\pi i} \int d^f q [G(\mathbf{q}, \mathbf{q}|E)^* - G(\mathbf{q}, \mathbf{q}|E)] , \quad (8)$$

and the bar stands for averaging over the random potential. We now make the assumption that, for a chaotic system, eq. (6) continues to hold if we first take the limit of small  $\hbar$ , and then disregard the random potential.

There are two different semiclassical approximations which can be used to compute the Green's function in the small- $\hbar$  limit [18,19]. The first applies when the distance between  $\mathbf{q}$  and  $\mathbf{q}'$  is small, in the sense that the classical path of least action with energy  $E$  which connects  $\mathbf{q}$  to  $\mathbf{q}'$  is well approximated by a linear function of time (in any coordinate system). This path then dominates the Green's function, and one finds

$$\bar{G}(\mathbf{q}', \mathbf{q}|E) = \int \frac{d^f p}{(2\pi\hbar)^f} e^{i\mathbf{p} \cdot (\mathbf{q}' - \mathbf{q})/\hbar} \frac{1}{E - H_W(\mathbf{p}, \bar{\mathbf{q}}) + i0^+} , \quad (9)$$

which immediately yields eq. (3) for  $C(\mathbf{q}', \mathbf{q}|E)$ . If, however, the classical path of least action is not well approximated by a linear function of time, and the value of this action is much larger than  $\hbar$ , then we have instead

$$\bar{G}(\mathbf{q}', \mathbf{q}|E) = \frac{1}{i\hbar(2\pi i\hbar)^{(f-1)/2}} \sum_{\text{paths}} |D_{\mathbf{p}}|^{1/2} e^{iS_{\mathbf{p}}/\hbar - i\nu_{\mathbf{p}}\pi/2} . \quad (10)$$

Here the sum is over all classical paths connecting  $\mathbf{q}$  to  $\mathbf{q}'$  with energy  $E$ , action

$$S_{\mathbf{p}} = \int_{\mathbf{q}}^{\mathbf{q}'} \mathbf{p} \cdot d\mathbf{q} , \quad (11)$$

focal point number  $\nu_p$ , and fluctuation determinant

$$D_p = \det \begin{pmatrix} \frac{\partial^2 S_p}{\partial \mathbf{q}_2 \partial \mathbf{q}_1} & \frac{\partial^2 S_p}{\partial E \partial \mathbf{q}_1} \\ \frac{\partial^2 S_p}{\partial \mathbf{q}_2 \partial E} & \frac{\partial^2 S_p}{\partial E^2} \end{pmatrix}. \quad (12)$$

Eqs. (6) and (10) give the correct formula for  $C(\mathbf{q}', \mathbf{q}|E)$  when  $|\mathbf{q}' - \mathbf{q}|$  is large [17].

This formula is most useful under circumstances where the sum over paths is dominated by a single path of least action. An interesting example is a two-dimensional billiard in a perpendicular magnetic field  $\mathbf{B}$ . In the billiard interior the hamiltonian is  $H = (\mathbf{p} - e\mathbf{A})^2/2m$ , and we will work in the gauge in which the vector potential is  $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{q}$ . Assuming that it is not blocked, the classical path of least action is a circular arc with length  $\ell$ , related to the separation  $L = |\mathbf{q}' - \mathbf{q}|$  and classical cyclotron radius  $R = (2mE)^{1/2}/|eB|$  via  $\ell = 2R \sin^{-1}(L/2R)$ . The action for this path can be divided into a geometric part and a gauge-dependent part,  $S_p = S_{\text{geom}} + S_{\text{gauge}}$ . The geometric part is

$$\begin{aligned} S_{\text{geom}} &= \hbar k \left( \ell - \frac{\mathcal{A}}{R} \right) \\ &= \hbar k L \left( 1 - \frac{L^2}{24R^2} + \dots \right), \end{aligned} \quad (13)$$

where  $\hbar k = (2mE)^{1/2}$ , and  $\mathcal{A} = \frac{1}{2}R\ell - \frac{1}{2}R^2 \sin(\ell/R)$  is the area enclosed by the circular arc and the straight line connecting  $\mathbf{q}$  to  $\mathbf{q}'$ . For our gauge choice, the gauge-dependent part is

$$S_{\text{gauge}} = \frac{1}{2}e\mathbf{B} \cdot (\mathbf{q} \times \mathbf{q}'). \quad (14)$$

In any gauge, the determinant  $|D_p|$  is given by

$$|D_p| = \frac{m^2}{\hbar k L} \left( 1 - \frac{L^2}{4R^2} \right)^{-1/2}. \quad (15)$$

Keeping only the contribution of this one path, and recalling that  $\bar{\rho}(E) = mA_B/2\pi\hbar^2$  for a two-dimensional billiard with area  $A_B$ , we find from eqs. (6) and (10) that

$$C(\mathbf{q}', \mathbf{q}|E) = A_B^{-1} \exp(iS_{\text{gauge}}/\hbar) \frac{\cos(S_{\text{geom}}/\hbar - \pi/4)}{(\pi k L/2)^{1/2} (1 - L^2/4R^2)^{1/4}}. \quad (16)$$

We need  $kL \gg 1$  for this formula to hold. Also, to avoid an integrable region of phase space, a circle of radius  $R$  must not be able to fit inside the billiard; this implies  $L < 2R$ .

We can compare eq. (16) with the result of using Berry's formula, eq. (3); with our gauge choice we find

$$C(\mathbf{q}', \mathbf{q}|E) = A_B^{-1} \exp(iS_{\text{gauge}}/\hbar) J_0(kL). \quad (17)$$

For large  $kL$ , we can use the asymptotic form of the Bessel function to get

$$C(\mathbf{q}', \mathbf{q}|E) = A_B^{-1} \exp(iS_{\text{gauge}}/\hbar) \frac{\cos(kL - \pi/4)}{(\pi k L/2)^{1/2}}. \quad (18)$$

We see that Berry's formula misses the corrections due to the finite cyclotron radius  $R$  which are present in eq. (16).

If we make a gauge transformation

$$\mathbf{A}(\mathbf{q}) \rightarrow \mathbf{A}(\mathbf{q}) + \nabla \Phi(\mathbf{q}) , \quad (19)$$

where  $\Phi(\mathbf{q})$  is any smooth function, then

$$S_{\text{gauge}} \rightarrow S_{\text{gauge}} + e\Phi(\mathbf{q}') - e\Phi(\mathbf{q}) , \quad (20)$$

and so eq. (16) implies

$$C(\mathbf{q}', \mathbf{q}|E) \rightarrow e^{+ie[\Phi(\mathbf{q}') - \Phi(\mathbf{q})]/\hbar} C(\mathbf{q}', \mathbf{q}|E) . \quad (21)$$

That this is correct can be seen by recalling that a wave function  $\psi(\mathbf{q})$  transforms as

$$\psi(\mathbf{q}) \rightarrow e^{+ie\Phi(\mathbf{q})/\hbar} \psi(\mathbf{q}) \quad (22)$$

under (19), and that  $C(\mathbf{q}', \mathbf{q}|E_\alpha)$  is the expected value of  $\psi_\alpha(\mathbf{q}')\psi_\alpha^*(\mathbf{q})$ . On the other hand, eq. (3) implies instead that

$$C(\mathbf{q}', \mathbf{q}|E) \rightarrow e^{+ie(\mathbf{q}' - \mathbf{q}) \cdot \nabla \Phi(\bar{\mathbf{q}})/\hbar} C(\mathbf{q}', \mathbf{q}|E) , \quad (23)$$

which again illustrates the fact that Berry's formula is valid only when  $|\mathbf{q}' - \mathbf{q}|$  is sufficiently small.

I now turn to another issue, the statistical properties of transition matrix elements  $A_{\alpha\beta} \equiv \langle \alpha | A | \beta \rangle$  in chaotic systems. Here the operator  $A$  is a smooth,  $\hbar$ -independent function of the coordinates and momenta, and  $|\alpha\rangle$  and  $|\beta\rangle$  are different energy eigenstates. We will be interested in the average value of  $|A_{\alpha\beta}|^2$  when the energies  $E_\alpha$  and  $E_\beta$  are each varied over a range which is small classically, but encompasses many quantum energy levels. For later convenience let us define

$$\bar{E} = \frac{1}{2}(E_\alpha + E_\beta) \quad \text{and} \quad \hbar\omega = E_\beta - E_\alpha . \quad (24)$$

We will do our computations in the limit  $\hbar \rightarrow 0$  with  $\bar{E}$  and  $\omega$  held fixed. The mean level spacing near energy  $\bar{E}$  is  $1/\bar{\rho}(\bar{E}) \sim \hbar^f$ , so our condition on the ranges of  $E_\alpha$  and  $E_\beta$  is easy to fulfill.

One approach [20–22] to this problem is to relate the average value  $\langle |A_{\alpha\beta}|^2 \rangle$  to the classical time-correlation function of the observable  $A$  via

$$\langle |A_{\alpha\beta}|^2 \rangle = \frac{1}{\tau_H} \int_{-\tau_H}^{\tau_H} dt e^{i\omega t} \int d\mu_{\bar{E}} A_W(\mathbf{p}_t, \mathbf{q}_t) A_W(\mathbf{p}, \mathbf{q}) . \quad (25)$$

Here  $\tau_H = 2\pi\hbar\bar{\rho}(\bar{E})$  is the Heisenberg time,  $(\mathbf{p}_t, \mathbf{q}_t)$  is the point in phase space which is reached classically at time  $t$  when starting from  $(\mathbf{p}, \mathbf{q})$  at time zero,  $A_W(\mathbf{p}, \mathbf{q})$  is the Weyl symbol of the operator  $A$ , and  $d\mu_E$  is the Liouville measure on the surface in phase space with energy  $E$ , given by

$$d\mu_E = \frac{1}{\bar{\rho}(E)} \frac{d^f p d^f q}{(2\pi\hbar)^f} \delta(E - H_W(\mathbf{p}, \mathbf{q})) . \quad (26)$$

Note that the notation, though useful later, is somewhat misleading:  $d\mu_E$  is a purely classical,  $\hbar$ -independent object. We will not discuss the derivation [20–22] of eq. (25) here; a brief and non-rigorous review is given elsewhere [23].

Another approach [4,24,7] to computing  $\langle |A_{\alpha\beta}|^2 \rangle$  is to write  $|A_{\alpha\beta}|^2$  in terms of the energy eigenfunctions  $\psi_\alpha(\mathbf{q})$  and  $\psi_\beta(\mathbf{q})$ , and then average  $|A_{\alpha\beta}|^2$  over the eigenfunction probability distribution of eq. (1). This calculation is simplified if  $A$  is a function of  $\mathbf{q}$  only (rather than both  $\mathbf{q}$  and  $\mathbf{p}$ ), and so we specialize to this case. We get

$$\langle |A_{\alpha\beta}|^2 \rangle = \int d^f q' d^f q C(\mathbf{q}, \mathbf{q}' | E_\alpha) A_W(\mathbf{q}') C(\mathbf{q}', \mathbf{q} | E_\beta) A_W(\mathbf{q}) . \quad (27)$$

It is not at all clear that eq. (27) gives the same result for  $\langle |A_{\alpha\beta}|^2 \rangle$  as eq. (25), an issue which was raised (in the context of the diagonal matrix elements) by Austin and Wilkinson [25]. If Berry's formula is used for  $C(\mathbf{q}', \mathbf{q} | E)$ , then eqs. (25) and (27) do not necessarily agree, as can be seen by working out some simple examples. In this case, it is eq. (27) which is wrong. The reason is that  $\mathbf{q}$  and  $\mathbf{q}'$  are independently integrated in eq. (27), and so  $|\mathbf{q}' - \mathbf{q}|$  is generically large. Therefore we should use eqs. (6) and (10) for  $C(\mathbf{q}', \mathbf{q} | E)$ , rather than eq. (3). If we do, then eqs. (25) and (27) give the same result for  $\langle |A_{\alpha\beta}|^2 \rangle$ .

To see this still requires some work. A detailed exposition is given elsewhere [23], and here we will just highlight the key elements. One is the diagonal approximation [26]; after substituting eqs. (6) and (10) into eq. (27), the double sum over paths is collapsed to a single sum, since the off-diagonal terms will have rapidly oscillating phases. In different but related contexts [26,24], this requires restricting the single sum to paths whose elapsed times are less than the Heisenberg time, and we will assume the same is true here. The action difference in the single sum is

$$\begin{aligned} S_{p_\beta} - S_{p_\alpha} &= (E_\beta - E_\alpha) \frac{\partial S_p}{\partial E} \Big|_{E=\bar{E}} + \dots \\ &= \hbar\omega\tau_p + O(\hbar^2) , \end{aligned} \quad (28)$$

where  $\tau_p$  is the elapsed time along the path. We assume that  $\nu_{p_\beta} = \nu_{p_\alpha}$ , since  $\nu_p$  is a topological quantity which in general will not change when the energy of the path is varied slightly. Finally, we note that the determinant  $D_p$  can be written as

$$D_p = \det \begin{pmatrix} -\frac{\partial \mathbf{p}}{\partial \mathbf{q}'} & \frac{\partial \tau}{\partial \mathbf{q}'} \\ -\frac{\partial \mathbf{p}}{\partial E} & \frac{\partial \tau}{\partial E} \end{pmatrix} . \quad (29)$$

Here  $\mathbf{p} = -\partial S_p / \partial \mathbf{q}$  is the momentum at the beginning of the path, and  $\tau = \tau_p$  is the elapsed time along the path. Eq. (29) shows us that  $|D_p|$  can be thought of [18] as a jacobian for a change of variables from the final position  $\mathbf{q}'$  and total energy  $\bar{E}$  to the initial momentum  $\mathbf{p}$  and elapsed time  $\tau$ . Eq. (27) already has an integral over  $\mathbf{q}'$ , and to get one over  $\bar{E}$  we insert  $1 = \int d\bar{E} \delta(\bar{E} - H_W(\mathbf{p}, \mathbf{q}))$ . Now we can make the change of integration variables suggested by eq. (29), and we find

$$\langle |A_{\alpha\beta}|^2 \rangle = \frac{1}{\pi \hbar \bar{\rho}(\bar{E})^2} \int_0^{\tau_H} d\tau \int \frac{d^f p d^f q}{(2\pi \hbar)^f} \sum_{\text{paths}} \delta(\bar{E} - H_W(\mathbf{p}, \mathbf{q})) \cos(\omega\tau) A_W(\mathbf{q}') A_W(\mathbf{q}) . \quad (30)$$

The sum is now over all paths which begin at  $(\mathbf{p}, \mathbf{q})$  and have elapsed time  $\tau$ . However, there is only one such path, and so the sum over paths may be dropped. Also,  $\mathbf{q}'$  is the position at time  $\tau$ , and it is now more properly denoted  $\mathbf{q}_\tau$ . Using eq. (26), the fact that time-translation invariance implies that  $\int d\mu_{\bar{E}} A_W(\mathbf{q}_\tau) A_W(\mathbf{q})$  is an even function of  $\tau$  (even if the system is not time-reversal invariant), and  $\tau_H = 2\pi\hbar\bar{\rho}(\bar{E})$ , we see that eq. (30) can be rewritten as

$$\langle |A_{\alpha\beta}|^2 \rangle = \frac{1}{\tau_H} \int_{-\tau_H}^{+\tau_H} d\tau e^{i\omega\tau} \int d\mu_{\bar{E}} A_W(\mathbf{q}_\tau) A_W(\mathbf{q}) , \quad (31)$$

which is equivalent to eq. (25).

Another quantity of interest is the size of the fluctuations in the diagonal matrix elements  $A_{\alpha\alpha}$ . If we first shift  $A$  (if necessary) so that  $\langle A_{\alpha\alpha} \rangle = \int d\mu_{E_\alpha} A_W(\mathbf{q}) = 0$ , then the object we wish to evaluate is  $\langle |A_{\alpha\alpha}|^2 \rangle$ . This has been done previously [27,28,24] by making use of the trace formula [18,29,30] and properties of periodic orbits. Here we will instead compute  $\langle |A_{\alpha\alpha}|^2 \rangle$  by averaging over the probability distribution for energy eigenfunctions [4,24,7,23]. In the case of a system which is not invariant under time reversal, the energy eigenfunctions are generically complex, and the relevant formula is [15]

$$\langle \psi_1^* \psi_2 \psi_3^* \psi_4 \rangle = \langle \psi_1^* \psi_2 \rangle \langle \psi_3^* \psi_4 \rangle + \langle \psi_1^* \psi_4 \rangle \langle \psi_3^* \psi_2 \rangle , \quad (32)$$

where  $\psi_i = \psi_\alpha(\mathbf{q}_i)$ . If the system is invariant under time reversal, the energy eigenfunctions are real, and we have instead [15]

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = \langle \psi_1 \psi_2 \rangle \langle \psi_3 \psi_4 \rangle + \langle \psi_1 \psi_4 \rangle \langle \psi_2 \psi_3 \rangle + \langle \psi_1 \psi_3 \rangle \langle \psi_2 \psi_4 \rangle . \quad (33)$$

Combined with the previous results for  $\langle |A_{\alpha\beta}|^2 \rangle$ , we find that

$$\langle |A_{\alpha\alpha}|^2 \rangle = \frac{2/\beta}{\tau_H} \int_{-\tau_H}^{\tau_H} d\tau \int d\mu_{E_\alpha} A_W(\mathbf{q}_\tau) A_W(\mathbf{q}) . \quad (34)$$

Here  $\beta = 1$  for a system which is invariant under time reversal, and  $\beta = 2$  for a system which is not. Eq. (34) is in agreement with the earlier results [27,28,24].

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